Quantum Stochastic Analysis in Banach space

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Summary

Setup and bullet notation

Setup

- $\mathfrak{X}$ *a fixed Banach space.*
- $\mathcal{F} = \Gamma(L^2(\mathbb{R}_+; \mathsf{k}))$ *for a fixed Hilbert space* k.
- $\mathbb{S} \subset L^2(\mathbb{R}_+; \mathsf{k})$ *the set of all* k*-valued step functions.*
- $\mathcal{E} = Lin\{\varepsilon(f) : f \in \mathbb{S}\}$.

Recall that $\mathcal{F} = \mathcal{F}_{[0,t)} \otimes \mathcal{F}_{[t,\infty)}$ and $\nabla_t \varepsilon(f) = f(t) \otimes \varepsilon(f) \in \mathsf{k} \otimes \mathcal{F}$.

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Notation

For $P \in B(\langle \mathcal{F}_{[0,t)}|; \mathcal{A})$ *and* $Q \in B(\langle \mathcal{F}_{[t,\infty)}|; \mathcal{A})$,

$$P \bullet Q := m \circ (P \widehat{\otimes} Q) \in B(\langle \mathcal{F}|; \mathcal{A}),$$

where m *is the operator* $\mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ *induced by multiplication in* $\mathcal{A}$, *using* $\langle \mathcal{F}_{[0,t)}| \widehat{\otimes} \langle \mathcal{F}_{[t,\infty)}| = \langle \mathcal{F}_{[0,t)} \otimes \mathcal{F}_{[t,\infty)}| = \langle \mathcal{F}|$.

CI identification

Natural CI isomorphisms

$\mathsf{U}, \mathsf{V}, \mathsf{W}$ concrete operator spaces
H Hilbert space

- $\mathsf{W} \otimes_{\mathsf{M}} |\mathsf{H}\rangle \cong CB(\langle \mathsf{H}|; \mathsf{W})$.
- $CB(\mathsf{U}; CB(\mathsf{V}; \mathsf{W})) \cong CB(\mathsf{V}; CB(\mathsf{U}; \mathsf{W}))$.

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Viewpoint on Standard Mapping Processes on A

$$(k_t)_{t \geq 0} \text{ in } CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\mathcal{F}))$$

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$$(k_t)_{t\geq 0} \text{ in } CB(\mathsf{A}; \mathsf{A} \otimes_{\mathsf{M}} B(\mathcal{F})) \subset \quad L\Big(\mathcal{E}; CB(\mathsf{A}; \mathsf{A} \otimes_{\mathsf{M}} |\mathcal{F}\rangle)\Big)$$

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QS Processes in Banach space

$$\text{Families } (m_t)_{t\geq 0} \text{ in } L(\mathcal{E}; B(\langle \mathcal{F}|; \mathfrak{X})).$$

Vector processes: $\varepsilon(f)$-adaptedness

Let $f \in L^2(\mathbb{R}_+; \mathsf{k})$.

Definition

A family $X = (X_t)_{t \geq 0}$ in $B(\langle \mathcal{F}|; \mathfrak{X})$ is an $\varepsilon(f)$-adapted vector process in $\mathfrak{X}$ if it satisfies

- $t \mapsto X_t(\langle \xi|)$ *weakly measurable.*
- $X_t = X(t) \widehat{\otimes} R\Big(|\varepsilon(f_{[t,\infty)})\rangle\Big)$, *where* $X(t) \in B(\langle \mathcal{F}_{[0,t)}|; \mathfrak{X})$.

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Proposition

For a family $X = (X_t)_{t\geq 0}$ in $B(\langle\mathcal{F}|; \mathfrak{X})$, TFAE:

1. *X is an $\varepsilon(f)$-adapted vector process in $\mathfrak{X}$.*
2. $\forall_{\omega\in\mathfrak{X}^*}$ $X^\omega := (X_t^\omega = \omega \circ X_t)_{t\geq 0}$ *defines a "standard" $\varepsilon(f)$-adapted vector process in $\langle\mathcal{F}|^* = \mathcal{F}$.*

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Now define $\mathcal{S}_t^{\mathfrak{X}} X$ in $L(\langle \mathcal{E}|; \mathfrak{X})$ by duality:

$$(\mathcal{S}_t^{\mathfrak{X}} X)(\langle \varepsilon|) := \int_0^t X_s\Big(\langle \nabla_s(\varepsilon)|\Big) \, \mathrm{d}s.$$

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Key fact

$$\omega\Big((\mathcal{S}_t^{\mathfrak{X}} X)(\langle \varepsilon|)\Big) = \langle \varepsilon, \mathcal{S}_t(X^\omega)\rangle \quad (\omega \in \mathfrak{X}^*).$$

Let $f \in L^2(\mathbb{R}_+; \mathsf{k})$.

Properties

Let $X = (X_t)_{t\geq 0}$ be a Skorohod-integrable $\varepsilon(f)$-adapted vector process. Then

1. $(\mathcal{S}_t^{\mathfrak{X}} X)_{t\geq 0}$ *defines an $\varepsilon(f)$-adapted vector process in $\mathfrak{X}$.*

Vector processes: $(\mathcal{S}_t^{\mathfrak{X}} X)_{t\geq 0}$ as $\varepsilon(f)$-adapted vector process

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1. $(\mathcal{S}_t^{\mathfrak{X}} X)_{t\geq 0}$ *defines an $\varepsilon(f)$-adapted vector process in $\mathfrak{X}$.*
2. $\|\mathcal{S}_t^{\mathfrak{X}} X - \mathcal{S}_r^{\mathfrak{X}} X\| \leq C_{f,[r,t)} \sup_{\omega \in \mathrm{Ball}[\mathfrak{X}^*]} \Big(\int_r^t \|\omega \circ X_s\|^2 \, \mathrm{d}s \Big)^{1/2}$.

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3. $\forall_{\omega \in \mathfrak{X}^*} \ \ \omega \circ (\mathcal{S}_t^{\mathfrak{X}} X) = \mathcal{S}_t(X^\omega)$.
4. *If X is locally bounded then $t \mapsto \mathcal{S}_t^{\mathfrak{X}} X$ is locally Hölder $1/2$-continuous.*

QS process in $\mathfrak{X}$

Definition

A family $m = (m_t)_{t\geq 0}$ *in* $L\Big(\mathcal{E}; B(\langle\mathcal{F}|; \mathfrak{X})\Big)$ *is a QS process in* $\mathfrak{X}$ *if it satisfies*

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2. $m_{t,\varepsilon(f)} = m^{\varepsilon(f)}(t)\widehat{\otimes}R\Big(|\varepsilon(f_{[t,\infty)})\rangle\Big)$,
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Example (QS Process in $B(\mathsf{h})$)

For a "standard" QS process $X = (X_t)_{t\geq 0}$ *in* $B(\mathsf{h}\otimes\mathcal{F})$.

$$m_{t,\varepsilon}(\langle\xi|) := (I_{\mathsf{h}} \otimes \langle\xi|)X_t(I_{\mathsf{h}} \otimes |\varepsilon\rangle)$$

defines a QS process in our (wider) sense.

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- *Associated semigroups* $\{P^{c,d} : c, d \in \mathsf{k}\}$ *of m:*

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$$P_t^{c,d} := m_{t,\varepsilon(d_{[0,t)})}(\langle\varepsilon(c_{[0,t)})|).$$
- *m is Markov-regular if each $P^{c,d}$ is norm continuous.*
- *m is adjointable if there is a QS cocycle $m^\dagger$ in $\mathcal{A}^\dagger$ satisfying*
$$m^\dagger_{t,\varepsilon}(\langle\varepsilon'|) = \Big(m_{t,\varepsilon'}(\langle\varepsilon|)\Big)^\dagger.$$

Set $\widehat{\mathsf{k}} = \mathbb{C} \oplus \mathsf{k}$

Theorem

For $\gamma \in L(\widehat{\mathsf{k}}; B(\langle\widehat{\mathsf{k}}|; \mathcal{A}))$, the QSDE

$$\mathrm{d}m_t = m_t.\gamma \, \mathrm{d}\Lambda(t), \quad m_{0,\varepsilon}(\langle\xi|) = \langle\xi, \varepsilon\rangle 1_{\mathcal{A}}$$

has a unique solution, denoted m^γ. It is given by a form of Picard iteration:

$$m^\gamma_{t,\varepsilon} = \sum_{n\geq 0} \Lambda^{(n)}_t (\gamma^{\bullet n})_\varepsilon \in B(\langle\mathcal{F}|; \mathcal{A}).$$

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Properties:

1. *$t \mapsto m^\gamma_{t,\varepsilon}$ is locally Hölder $1/2$-continuous.*
2. *m^γ is a Markov-regular QS cocycle.*
3. *If γ is adjointable then m^γ is also adjointable and*
$$(m^\gamma)^\dagger = m^{\gamma^\dagger}$$

Stochastic generators for QS cocycles

Theorem

Let m be an adjointable, Markov-regular QS cocycle in $\mathcal{A}$ such that

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Then there is $\gamma \in L(\widehat{\mathsf{k}}; B(\langle\widehat{\mathsf{k}}|; \mathcal{A}))$ such that

$$m = m^{\gamma}.$$

Idea of the Proof

1. For fixed $w \in \mathbb{C}, d \in \mathsf{k}$, define
$$\gamma_1 \begin{pmatrix} w \\ d \end{pmatrix} \in B(\langle\mathbb{C}|; \mathcal{A}) \text{ and } \gamma_2 \begin{pmatrix} w \\ d \end{pmatrix} \in B(\langle\mathsf{k}|; \mathcal{A}) \text{ by}$$

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$$\gamma_1 \begin{pmatrix} w \\ d \end{pmatrix} : \langle z| \mapsto \overline{z}\Big(\beta_{0,d} + (w-1)\beta_{0,0}\Big), \textit{ and}$$
$$\gamma_2 \begin{pmatrix} w \\ d \end{pmatrix} := \mathrm{st.}\lim_{t\to 0} \frac{1}{\sqrt{t}}\Big(m_{t,\varepsilon(d_{[0,t)})} - m_{0,\varepsilon(d_{[0,t)})}\Big) \circ E_t$$
$$+ (w-1)\,\mathrm{st.}\lim_{t\to 0} \frac{1}{\sqrt{t}}\Big(m_{t,\varepsilon(0)} - m_{0,\varepsilon(0)}\Big) \circ E_t,$$
where E_t is the isometry $\langle c| \in \langle \mathsf{k}| \mapsto \frac{1}{\sqrt{t}}\langle c_{[0,t)}| \in \langle \mathcal{F}|$.
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$$\gamma(\eta) = [\gamma_1(\eta) \;\; \gamma_2(\eta)] \in B(\langle \widehat{\mathsf{k}}|; \mathcal{A}) \quad (\eta \in \widehat{\mathsf{k}}).$$

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$$\gamma(\eta) = [\gamma_1(\eta) \;\; \gamma_2(\eta)] \in B(\langle \widehat{\mathsf{k}}|; \mathcal{A}) \quad (\eta \in \widehat{\mathsf{k}}).$$

References

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